Optimum Sectional Shape of Dominant Mode Waveguide

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***Abstract*—Approximating the cross section of a waveguide by a truncated Fourier series and using the finite element method (FEM), together with the quasi-Newton optimization method, the optimum cross section of the dominant mode waveguide which has minimum conductor loss is obtained.**

We take the attenuation constant at the cutoff frequency of the second higher mode as the index of good quality. This index simplifies the computation and gives a unique solution.

The obtained optimum cross section is a kind of cigar shape. The Fourier series converges quite quickly, supporting the reliability of the numerical results. This optimum cross section gives 9.4-percent smaller conductor loss than, and the same frequency bandwidth as, the standard rectangular cross section. The theoretical results are confirmed by measurements.

I. Introduction

IN THE DESIGN of microwave communication systems, the requirements for the bandwidth and attenuation of the antenna feeders are quite stringent. While rectangular or elliptical waveguides are commonly used for rather long transmission, there has been no sign indicating the optimality of these standard waveguides.

Though some investigations on a specific sectional shape were made [1], there has not been published any report on optimality concerning general sectional shapes. Recently, numerical analysis using the finite element method (FEM) has evolved, and the transmission characteristics of waveguides of general sectional shape have been analyzed numerically [2]–[4], but no solution has been reported for the optimization of loss characteristics of a dominant mode waveguide.

In the present paper, using the truncated Fourier series as the trial function for the sectional shape, together with the FEM and quasi-Newton optimization method, we have obtained the optimum cross section of a dominant mode waveguide which gives minimum conductor loss among other shapes.

II. Relative Attenuation Factor

As an index for comparison of the dominant mode waveguide attenuation characteristics for an arbitrary shape, a definition is given to a relative attenuation factor M. This factor is the dominant mode attenuation constant for a prescribed shape at the second mode cutoff frequency. Assuming the lowest transmission mode of the waveguide as an H mode, the relative attenuation factor of the mode is given by the following equation [5]:

$$\alpha \cdot Kc_2^{-1}(\text{neper}) = \sqrt{\frac{Kc_2}{8\zeta_0\sigma}} \cdot U(P) \tag{1}$$

where

$$U(P) = \frac{\sqrt{P}}{Kc_2} \cdot \sqrt{1-\left(\frac{Kc_1}{Kc_2}\cdot\frac{1}{P}\right)^2} \cdot \left[\frac{1}{Kc_1^2}\cdot A + \frac{1}{\left(\frac{Kc_2}{Kc_1}\right)^2 \cdot P^2 - 1}\cdot B\right] \tag{2}$$

ζ_0 intrinsic impedance of free space (ohms),
σ conductivity of the conductor wall (Siemens),
P variable normalized with second mode cutoff frequency, $f_0/fc_2 = K_0/Kc_2$

$$A = \frac{\int\left(\frac{\partial\phi}{\partial s}\right)^2 ds}{\int\int \phi^2\, dS} \qquad B = \frac{\int \phi^2\, ds}{\int\int \phi^2\, dS}. \tag{3}$$

The coefficients Kc_1 and Kc_2 are the lowest and the secondary eigenvalues of the following Helmholz equation, over the cross section S and with Neumann boundary conditions

$$\nabla^2\phi + Kc^2\phi = 0 \qquad \frac{\partial\phi}{\partial n} = 0. \tag{4}$$

In this case, ds is the line element of the wall cross section and the dimensionality of A and B is $[1/m^3]$ and $[1/m]$, respectively.

Furthermore, (2) is reduced to the following form which represents the conductor loss α at the secondary mode's cutoff frequency by putting $P=1$:

$$M = U(P=1) = \sqrt{\frac{Kc_2^2 - Kc_1^2}{Kc_2^2}} \cdot \left[\frac{1}{Kc_1^2}\cdot A + \frac{Kc_1^2}{Kc_2^2 - Kc_1^2}\cdot B\right]. \tag{5}$$

The value of M is calculated analytically for typical shapes. We get 0.920 for an optimized rectangle with an axis ratio of 1:2, and 1.11 for an ellipse optimized with an eccentricity of 0.61. Namely, the ellipse is about 21-percent larger than the rectangle in the M value.

When the attenuation characteristic of optimized rectangles and ellipses is calculated by (2), it shows a decreasing

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tendency between the first and second cutoff frequencies. Namely, in the interval between the first and second cutoff frequencies, that represents the dominant mode transmission bandwidth, the minimum value of $U(p)$ is given by M. This is not necessarily true for a single-mode broad-band waveguide like a ridge waveguide. It can be shown, however, that within the range of a shape parameter representing a ridge, the minimum attenuation value within the dominant mode transmission band is always larger than the value of M for the optimized shape parameter. Therefore, for any cross-sectional configuration, the relative attenuation factor M can be found, and the shape for minimum attenuation is determined.

III. Trial Function of Cross-Sectional Configuration

As a function for expressing an arbitrary cross-sectional configuration, a finite Fourier series may be proposed. The Fourier series, though simple in the function form, has a defect in that it cannot represent a nonconvex shape. As a function capable of nonconvex representation, a spline-cubic approximation can be cited. In the nonconvex shape, however, the number of configuration parameters increases, thus making the calculation process cumbersome. Therefore, while the calculation by the spline-curve is a rough approximation, if it is found that that the configuration to make M smaller can be expressed by a Fourier series, then the Fourier series alone will do [6].

As shown in Fig. 1, expressing the cross-sectional geometry by a polar coordinate representation $r = R(\theta)$, we get an equation with five Fourier coefficients

$$R(\theta) = a_0 + \sum_{n=1}^{4} a_n \cos 2n\theta. \tag{6}$$

Equation (6) means that the angle of intersection of the symmetric axes of cross section is perpendicular. Selection of the number of series of Fourier coefficients is made by how close the shape of the cross-sectional element for FEM can approach the configuration expressed by the Fourier expansion.

Because of the limitation of computers' memories, the truncated number of the Fourier series should be restricted.

However, the adequacy of the Fourier series as a trial function can be confirmed by evaluating the convergence of the series in the optimized configuration.

The FEM is applied in the four-division domain shown in Fig. 1. In order to obtain a good approximation for the curve, six-node isoparametric curved elements are used [7]. The number of division is 36. The boundary conditions are the Neumann condition on the curve, and a combination of Dirichlet and Neumann conditions on the X-axes and Y-axes.

The relative attenuation factor M is determined by the FEM according to the automatic sub-division program of the domain by giving Fourier coefficients $a_0, \cdots, a_4$. In order to avoid the similar-figure condition, any one of the coefficients being fixed, the number of variables becomes 4. For these 4 variables, a group of eigenvalues is calculated, the lowest and second eigenvalues are selected, and further the lowest eigenfunction is calculated. As the method of obtaining the minimized M, a quasi-Newton method known as the multivariable optimizing numerical solution is adopted [8].

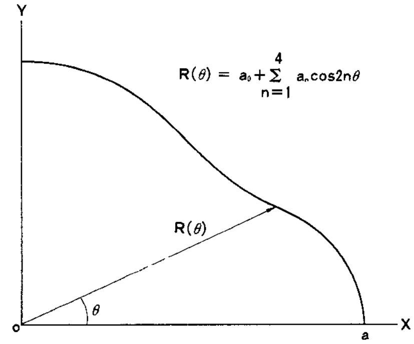


Fig. 1. Approximate function for waveguide cross-sectional view.

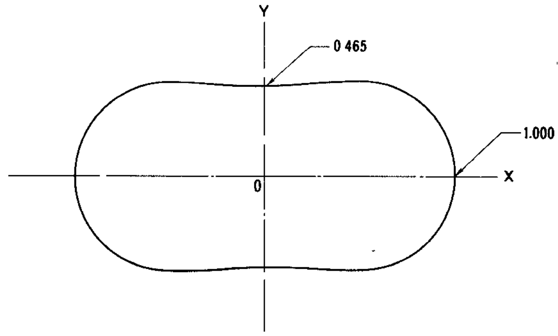


Fig. 2. Optimized configuration of waveguide cross section for dominant mode with Fourier coefficients $a_0 = 0.7201$, $a_1 = 0.2786$, $a_2 = 0.0171$, $a_3 = 0.0114$, $a_4 = -0.0044$. Eigenvalues: $Kc_1^2 = 2.94$, $Kc_2^2 = 11.68$.

IV. Optimized Configuration and its Property

The Fourier coefficients obtained pursuant to the optimizing calculation process described in Sections II and III are, respectively

$$a_0 = 0.7201 \quad a_1 = 0.2786 \quad a_2 = 0.0171$$
$$a_3 = -0.0114 \quad a_4 = -0.0044.$$

Rewritting these values by normalizing with the second cutoff wavelength λc_2, we get

$$R_0(\theta)\cdot\frac{1}{\lambda_{c2}} = 0.392 + 0.152\cos 2\theta + 0.009\cos 4\theta - 0.006\cos 6\theta - 0.002\cos 8\theta \tag{7}$$

and

$$M_0 = 0.833 \tag{8}$$

$$A_0 a^3 = 5.978 \quad B_0 a = 3.772 \tag{9}$$

$$K_{c10}\cdot a = 1.716 \quad K_{c20}\cdot a = 3.418 \qquad (10)$$
$$K_{c10} = 2\pi/\lambda_{c10} \quad K_{c20} = 2\pi/\lambda_{c20}$$

where

a half length of the longer side on the X-axes,
λ_{c10} dominants mode's cutoff wavelength,
λ_{c20} second mode's cutoff wavelength.

The following are the characteristic phenomena found out numerically under the optimized conditions.

1) The Fourier series selected as a configuration function has a good convergence, with the first and second items dominating. The third and fourth coefficients are less than 3 percent in terms of amplitude. Thus, the adequacy of representation by means of the Fourier series can be verified.

2) The cutoff frequency ratio between the dominant mode and higher modes K_{c20}/K_{c10} becomes nearly 2.0, which is equal to the maximum bandwidth of a rectangular waveguide. This means that the optimized configuration not only enjoys a minimum attenuation, but it also compares favorably in the bandwidth likewise with the rectangular configuration.

3) The second modes—the H_{01} mode (according to the rectangular mode representation) and H_{20} mode—are equal in eigenvalues, and accordingly they are in degenerated conditions. The conditions (2) and (3) are the results deduced from a numerical calculation, and are not analytical solutions. Since this is a special situation as a stationary condition, it is considered to provide a clue to approaching the analytical solution.

4) When the minimum value of the relative attenuation coefficient M is compared with that of other typical configurations, it is 9.4 percent smaller than a rectangle and 25 percent smaller than an ellipse.

5) In the circumferential length of cross sections, it is 8 percent less than a rectangle, and 2 percent less than an ellipse.

6) The minimum value of $U(P)$ occurs when $P = 1.17$, and it is 0.824. This result does not conflict with the assumption of calculation that the minimum value exists in a higher stage than the second mode cutoff frequency.

V. Optimized Cigar Shape

From the previous sections, it is known that the characteristics of the optimized configuration expressed by a Fourier series are not only the minimized attenuation of the dominant mode, but also its broad bandwidth and its short conductor circumferential length, i.e., a merit of less material requirement.

Also, it was learned that this configuration has a smooth and even appearance, showing rather a cigar-like shape.

Based on the foregoing knowledge, an approximation to the optimized configuration is conceived of a simple cigar construction which is made up of straight sides and semicircles, and is useful in both manufacture and inspection. This cigar shape involves a configuration parameter of a mere axis ratio, and is easier to find the optimized conditions by a numerical calculation than the case of approximation with the Fourier series.

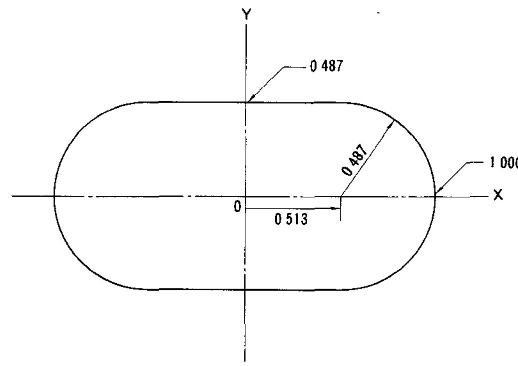


Fig. 3. Optimized cigar shape and its parameters. Eigenvalues: $Kc_1^2 = 3.02$, $Kc_2^2 = 11.63$.

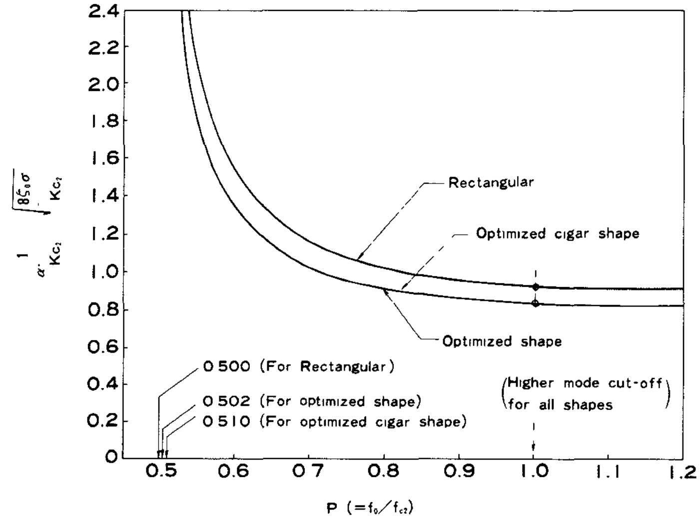


Fig. 4. Comparison of attenuation-frequency characteristics normalized with second mode's cutoff frequencies.

The results of this calculation are shown in Fig. 3, the ratio of longer axis to shorter axis being 0.487. The value of M at this is

$$M\,(\text{optimized cigar}) = 0.840. \qquad (11)$$

Let the length of the longer side be a, then

$$K_{c1}\cdot a = 1.74 \qquad K_{c2}\cdot a = 3.41 \qquad (12)$$

$$A\cdot a^3 = 6.04 \qquad B\cdot a = 3.79. \qquad (13)$$

The attenuation characteristic of this cigar shape differs from that of the optimized configuration by the Fourier series representation merely in its transmission bandwidth ratio of 1.96, which is 2 percent narrower than that of the optimized condition 2.00.

Fig. 4 shows the numerical calculation results of attenuation-frequency characteristics for both cases.

VI. Comparison With Measurement

In order to examine the accuracy of the numerical calculation concerning the optimized configuration, the attenuation-frequency characteristics and cutoff frequencies are measured on both rectangular and optimized cigar shapes, which are made by press-drawing the same material under the same manufacturing process. The difference in attenuation constant of around 10 percent, though important in practice, poses a rather troublesome problem to confirm by measurement. The rectangular waveguide is designed for a construction confirming to the IEC standard WR-159, with dimensions; ID (40±0.1 mm)×(20±0.1 mm), and wall thickness 1.6 mm, and 30 m-long are prepared. The optimized cigar shape is of (ID longer axis 43.2±0.1 mm(×(ID shorter axis 21.04±0.1))), and the wall thickness and unit length are the same as the rectangular one. And the second mode cutoff frequencies of both are designed for 7.50 GHz. The conductor material is taken from the same lot of deoxidized phosphorus copper consisting of 99.9 percent Cu or more, 0.04 percent P or less.

Fig. 5 shows the photos of both waveguides with flanges. The attenuation of 30 m-long samples is measured by a attenuator substitution method. To curb the effect of reflection, taper-transformers for the rectangular/cigar shape are prepared. This was made by a high-precision electroforming process.

The accuracy of measurement by the substitution process is ±0.05 dB, which accounts for about 3 percent of the approximate sample loss of 1.5 dB. However, repetitive measurements on many frequencies have brought about an accuracy improvement, so the error in this measurement may be said to be less than 1 percent.

The cutoff frequency is measured in the cavity resonator method. The separation of the higher modes from the dominant mode is achieved by changing the position and inclination angle of the coupling slit provided on the waveguide shorting plate. The length l of the waveguide for the cavity is 300 mm.

The cutoff frequencies are calculated from the observed resonant frequency f_{rn} by the following equation:

$$f_c = \sqrt{f_{rn}^2 - \left(\frac{nC_0}{2l}\right)^2} \qquad (14)$$

where l is the length of cavity, n is the number of degree of resonance mode, and C_0 is the velocity of light.

Because the accuracy of the sample length l, as well as that of the frequency measurement, can be easily kept 0.5 percent or less, the accuracy of the cutoff frequencies should be better than that of the attenuation measurement. Thus the measured attenuation versus frequency characteristics and the cutoff frequencies of rectangular and cigar shapes are shown in Fig. 6. The measured frequency band regarding the attenuation lies between 5.8 ~ 7.6 GHz, which approaches the dominant mode transmission band.

The calculated values are shown by full lines as extensions to Fig. 4. The conductivity of σ for a 100-percent pure copper value of 5.92×10^7 mho/m in the dc range is adopted.

The relative differences between rectangular and cigar shapes are: against the calculated value of 9.4 percent at the second cutoff frequency of 7.5 GHz, the measured value is 8.5 percent; and similarly, against 10.5 percent at

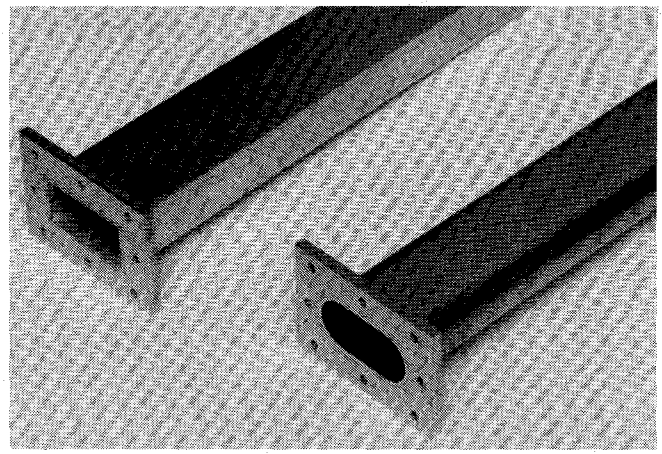
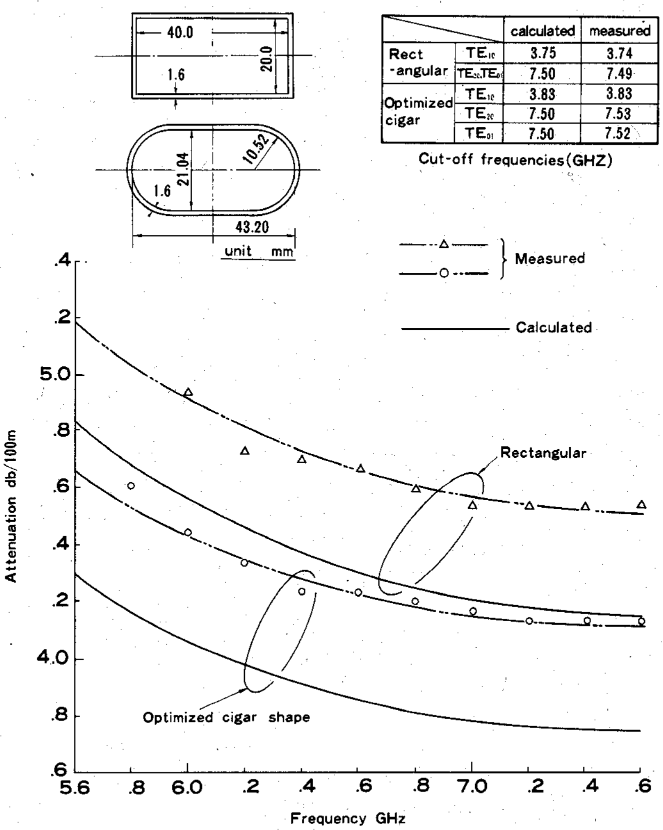


Fig. 5. Rectangular and optimized cigar-shaped waveguides with flanges.



		calculated	measured
Rect-angular	TE_{10}	3.75	3.74
	TE_{20}, TE_{01}	7.50	7.49
Optimized cigar	TE_{10}	3.83	3.83
	TE_{20}	7.50	7.53
	TE_{01}	7.50	7.52

Cut-off frequencies(GHZ)

Fig. 6. Attenuation characteristics and cutoff frequencies.

6.0 GHz, is 9.2 percent, respectively.

This good agreement of relative values over a wide frequency range can be easily deduced by taking $K_{c2}/K_{c1} = 1.96$ in (2) and by using, as A, B values, the values from (13) for cigar shapes, and the analytical solution for rectangular shapes.

The following equation will be found to be useful as an equation for expressing the ratio of losses of rectangular and cigar shapes:

$$U\,(\text{optimized cigar}) = \frac{1}{\pi\sqrt{P}} \cdot \frac{3.61P^2 + 0.82}{\sqrt{3.84\,P^2 - 1}} \qquad (P > 0.51) \qquad (15)$$

$$U(\text{rectangular}) = \frac{1}{\pi\sqrt{P}} \cdot \frac{4P^2+1}{\sqrt{4P^2-1}} \quad (P > 0.5) \quad (16)$$

so

$$\frac{U(\text{optimized cigar})}{U(\text{rectangular})} = \sqrt{\frac{4P^2-1}{3.84P^2-1}} \cdot \left[\frac{3.61P^2+0.842}{4P^2+1}\right] \quad (P > 0.51). \quad (17)$$

Namely, the attenuation-frequency characteristics of the optimized cigar shape having the second mode's cutoff frequency equal to the rectangular waveguide of axis ratio 2:1 can be obtained by multiplying (17) by a coefficient for the attenuation characteristics of the rectangular waveguide.

The differences between the calculated values and measured values can be understood by taking the dc conductivity as an effective conductivity in the microwave band.

The loss increment of about 9 percent of the measured values relative to the calculated values over a wide frequency band means the reduction of the effective conductivity of around 20 percent in the microwave band.

If the effective conductivity is taken as 4.9×10^7 mhos/m for both rectangular and cigar shapes, the measured values and calculated values will show a close agreement as shown by dotted lines.

The measured values of cutoff wavelengths show a close agreement with the calculated values by a deviation of less than 0.5 percent for both rectangular and cigar shapes. A slight deviation of two higher mode cutoff frequencies of cigar shape is considered to be within the manufacturing tolerance.

VII. Conclusion

Adoption of a coefficient with which to make a relative comparison of the dominant mode transmission loss with respect to an arbitrary cross-sectional shape has made it possible to find a optimized configuration having minimum loss.

The accuracy of FEM and the numerical optimization by the quasi-Newton Method as adopted has proved to be high enough to evaluate the improvement effects of the characteristics.

The optimized shape of the cross section can be expressed by a Fourier series for the polar coordinate with good convergence. Furthermore, it can be approximated by a cigar shape with an axis ratio which consists of a combination of semicircles and rectangular shapes.

Compared with the standard rectangular waveguide of axis ratio 2:1, it is more than 9.4 percent smaller in the attenuation, and the same in the dominant mode transmission bandwidth. Besides, several interesting properties have been found. In order to ascertain the accuracy of the foregoing numerical analytical method and the calculated values, measurements have been made of the attenuation frequency characteristics and cutoff frequencies of rectangular and cigar shapes, which are carefully made of the same material by the same manufacturing process. For an attenuation calculated value of 9.4 percent, a value of above 8.5 percent is registered by an actual measurement, and for the higher mode cutoff frequencies, an agreement between calculated and measured values is obtained within the manufacturing tolerance. These results are considered to be useful in applications such as long-length antenna feeders, high-power waveguides, and high-Q microwave circuit components.

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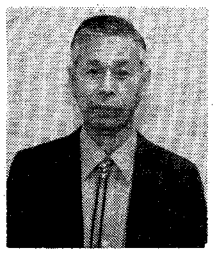

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Short Papers

An Accurate Approximation of the Impedance of a Circular Cylinder Concentric with an External Square Tube

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Abstract **—The problem of determining the characteristic impedance of a concentric coaxial transmission line having a circular inner conductor and a square outer conductor is reexamined. The Green's function for a rectangle is used to determine the geometrical capacitance of a series of structures ranging from 1–46 Ω with an error less than 10^{-5}. The method of analysis is illustrated in detail for the 1-Ω case. The results are presented in terms of the "outer shield factor" R_{eff}, which is defined as the ratio of the diameter of an outer circle, having the same capacitance as the outer square, to the side of the outer square. Values of this ratio are tabulated for impedances ranging from 1–46 Ω. These values are also plotted on a curve which can be read with an error of the order of 0.02 Ω for impedances greater than 3 Ω.**

I. Introduction

The determination of the characteristic impedance of the concentric coaxial line in which the outer conductor is a square and the inner conductor is a circle has been the subject of numerous treatments [1]–[16] appearing during the past forty years. In his discussion of this problem, Cohn [10] suggested that additional data between 30 and 2 Ω would be useful. This short paper provides this information.

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The treatment of this problem by Frankel [1], Oberhettinger and Magnus [2, pp. 75–78], and later by Laura and Luisoni [13], [14], is one in which the potential problem is solved exactly in a doubly connected region in which the outer conductor is a square, while the inner conductor is a four-lobed curve which approaches a circle ever more closely as its size decreases. Each circle internal to and concentric with the square has the same potential at eight equi-spaced points on its circumference. This potential function, except for an additive constant, is the Green's function [2, p. 36] for the square which has a logarithmic singularity at its center.

In this paper, a potential function is constructed, which is nearly constant on the outer circumference of the inner conductor, by suitably combining a number of Green's functions for the outer square whose logarithmic singularities are all inside of this circle.

II. The Green's Function

Fig. 1 shows an infinite lattice of positive and negative line charges whose logarithmic potential is zero along the boundary of a rectangle of width $2a$ and height $2b$ centered at the origin. This follows from the fact that, for every negative charge on one side of a boundary, there is an equal positive charge mirrored in it on the opposite side.

Consider for a moment the point Z' which is inside the rectangle in question. It determines a doubly infinite lattice of line charges which differ from it in location by integral multiples of $4a$ in the horizontal direction and by integral multiples of $4b$ in the vertical direction. Similar remarks can be made about the other three line charges shown in the upper right-hand quadrant

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